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1993 J. Phys. A: Math. Gen. 26 5551

(http://iopscience.iop.org/0305-4470/26/20/036)

J. Phys. A: Math. Gen. 26 (1993) 5551–5567. Printed in the UK

Non-cyclic geometric phases in a proposed two-photon interferometric experiment

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Received 5 January 1993, in final form 7 June 1993

Abstract. An experimental arrangement is proposed which combines the idea of two-photon interferometry with that of geometric phases. A pair of photons generated by parametric down-conversion is fed into a pair of Mach–Zehnder interferometers, in each of which there is phase shift produced by the helically wound optical fibre in the manner of Tomita and Chiao. If the winding angle is a multiple of 2π there is a closed path in parameter space, in accordance with the pioneering analysis of Berry. More general windings are also considered in which the path in parameter space is not closed, and the geometric phase for such open paths is calculated in accordance with the gauge-invariant prescription of Aitchison and Wanelik. If the proposed experiment is realized, it will provide a test for the non-cyclic geometric phase for photons and for the set of prescriptions equivalent to that of Aitchison and Wanelik. Furthermore, the geometric phase in this arrangement will be unequivocally quantum mechanical because of the non-classical character of two-photon interferometry.

1. Introduction

The path-dependent geometric phase factor in quantum mechanics was anticipated by Pancharatnam [1] and by Mead and Truhlar [2], among many others [3], and discovered by Berry [4] in the context of adiabatically changing environments of physical systems as in the celebrated Born–Oppenheimer treatment of the entanglement of the electronic and nuclear variables in molecules. In this context it is a quantal phase associated with the stationary states of a system with slow and *cyclic* variation of its environment represented by *classical* parameters in the Hamiltonian governing the system. Aharonov and Anandan [5], subsequently, reformulated and generalized Berry's result by disregarding the classical parameter space and considering non-adiabatic cyclic paths in the projective space of one-dimensional subspaces or rays of an appropriate Hilbert space. Using the non-trivial topological structure [6] of the projective Hilbert space, which has a mathematically natural Abelian connection, they showed that the geometric phase is an anholonomy for a parallel transported state-vector along an arbitrary closed path in the projective space, and that the geometric phase is independent of the parameterization of this path and of the means (i.e. the Hamiltonian used) by which the transport about this path is achieved. Their method of obtaining the geometric phase is completely quantum mechanical and purely geometrical in that one does not need the *classical* parameters required by Berry to represent the variations of the environment. Furthermore, it can be shown (at least in some simple models) that the Berry phase can be recovered from the purely quantum mechanical Aharonov–Anandan phase in an appropriate correspondence limit [7–9].

The geometric phase as defined by Aharonov and Anandan is a quantal phase associated with a cyclic evolution of a *pure* quantum mechanical state. If the system of interest is a subsystem of a larger composite system then it is not in a definite quantum state, and hence a ray or a state-vector can not be assigned to it. The total system may be described by an entangled state-vector, which evolves dynamically in accordance with the Schrödinger equation. In general, the dynamics of a subsystem can be obtained only by forming a statistical operator corresponding to the wavevector of the total system and then tracing out the variables of the remainder of the total system. Furthermore, non-local correlation *with no classical analogue* can occur between the subsystem and the remainder of the total system [10]. Consequently, a geometric phase associated with a subsystem of a total system in an entangled state can not be understood classically.

Tomita and Chiao [11] were the first to exhibit a geometric phase associated with a linearly polarized photon propagating in a helically coiled optical fibre. Berry [12] and others [13] raised the question that this geometric phase might be interpreted classically. To answer this objection, Kwiat and Chiao [14] observed the geometric phase at the quantum mechanical level by using an entangled pair of photons. They prepared incident light in an energy-entangled state of a pair of photons (signal and idler) by means of parametric fluorescence in a nonlinear optical crystal. Individually, the photon energies in this light were broad in spectrum, but they summed up to a sharp energy value because the pair was produced from a single photon with a sharp energy value. This entangled state is written as

$$|\Psi\rangle_{\text{in}}=\int \mathrm{d}E'\; A(E')|1\rangle_{E'}|1\rangle_{E-E'} \tag{1.1}$$

where $A(E')=A(E-E')$ is the complex probability amplitude for finding one photon with energy E' in the $n=1$ Fock state $|1\rangle_{E'}$, and another photon with energy $E-E'$ in the $n=1$ Fock state $|1\rangle_{E-E'}$. By employing a phase matching technique the two beams of photons were ensured to be horizontally polarized, and then coincidences in the detection of conjugate photons were observed. The photon pair is then fed into a Michelson interferometer in which one member of each pair acquired a Pancharatnam's phase [1, 15] (one form of Berry's phase) due to a cycle in polarization states. The outgoing state of the light from the Michelson interferometer is written as

$$|\Psi\rangle_{\text{out}}=\frac{1}{\sqrt{2}}\int \mathrm{d}E'\; A(E')|1\rangle_{E'}|1\rangle_{E-E'}[1+\exp\{\mathrm{i}\phi(E-E')\}] \tag{1.2}$$

where

$$\phi(E-E')\equiv 2\pi\,\frac{\Delta L}{\lambda_{E-E'}}+\phi_{\text{Berry}}$$

is the phase shift arising from the optical path difference ΔL of the interferometer for the photon with energy $E-E'$, plus the Berry's phase contribution for this photon. The coincidence rate between various detectors is then proportional to the probability of simultaneously finding one photon at a detector placed at position r_1 and another photon at a detector placed at position r_2. If a narrow-band filter centered at energy E' is placed in front of the detector at r_1, the coincidence rate becomes proportional to

$$|\Psi'_{\text{out}}(r_1, r_2, t)|^2=|\langle r_1, r_2, t|\Psi\rangle'_{\text{out}}|^2\propto 1+\cos\phi \tag{1.3}$$

where the prime denotes the output state after a von Neumann projection onto the eigenstate associated with the sharp energy E' upon measurement. Consequently, the phase ϕ is determined at the sharp energy $E-E'$. Kwiat and Chiao conclude: 'In the light of the observed violations of Bell's inequalities [16], it is incorrect to interpret these results in terms of an ensemble of conjugate signal and idler photons which possess definite, but unknown, conjugate energies before filtering and detection. Any observable, e.g. energy or momentum, should not be viewed as a local, realistic property carried by the photon *before it is actually measured.*' This clearly answers Berry's question regarding the necessity of the quantum description to explain the geometric phase effects associated with the optical rotations of photon fields.

The experiment that we propose in this paper also uses a pair of entangled photons, but with entanglement of their linear momenta rather than their energies, in accordance with the two-photon interferometric arrangements of Mandel's school [17], of Horne, Shimony, and Zeilinger (hereafter abbreviated HSZ) [18], of Rarity and Tapster [16], and others. Our proposed experiment also aims at greater generalization than that of Kwiat and Chiao, because the geometric phases which it exhibits may be non-cyclical as well as cyclical, that is, they may be due to open as well as to closed paths in parameter space, and consequently in projective Hilbert space. In the experiment of Kwiat and Chiao there is a closed path consisting of a cycle in the polarization states of the signal photon: from linear polarization to circular polarization and then back to linear polarization (at the initial point on the Poincaré sphere). They need to close the cycle, because interference fringes are not visible unless the initial and final polarizations are parallel. *As will be seen below, our proposed setup avoids the need for a* closed path.

In section 2 we shall review the notion of geometric phases, and summarize the manifestly gauge-invariant prescription of Aitchison and Wanelik for computing the geometric phase associated with a general path, open or closed. We shall note the virtues of their prescription, particularly the fact that the phase is a function of a path in the projective Hilbert space (i.e. the space of rays of a Hilbert space) rather than of a path in the Hilbert space itself, and we shall briefly note the relation of their prescription to earlier proposals. Section 3 will present the Tomita–Chiao setup and will outline derivation of the geometric phase in this setup, both in the cyclic and the non-cyclic case. Section 4 will give a brief review of the relevant parts of two-photon interferometry.

Section 5 is the core of the paper. In it we propose the incorporation of a pair of *Tomita–Chiao helical optical fibres as phase shifters into a two-photon interferometric* arrangement. The sinusoidal dependence of the coincidence counts upon the geometric phases is explicitly calculated in both the cyclic and non–cyclic cases.

Section 6 is a discussion of the non-classical character of this sinusoidal dependence ('two-photon fringes') and of the experimental testability of the non–cyclic geometric phase for photons obtained by a strictly quantum mechanical analysis.

Some detailed calculations of the cyclic and non-cyclic geometric phases are rendered in the appendix.

2. Cyclic and non-cyclic geometrical phases

In Berry's pioneering work on the geometric phase [4] a time-dependent Hamiltonian $H(t)$ was considered, with a period τ, i.e.

$$H(t+\tau)=H(t). \tag{2.1}$$

(In examples the Hamiltonian depends upon parameters, like components of a rotating magnetic field, and the parameters change with time satisfying (2.1).) Given an initial state-vector $|\psi(0)\rangle$ in the Hilbert space, the time-dependent state-vector $|\psi(t)\rangle$ can be obtained by solving the time-dependent Schrödinger equation with the specified initial condition. If the Hamiltonian changes adiabatically, and $|n(t)\rangle$ is a set of basis vectors of H(t)

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle \tag{2.2}$$

where $E_n(t)$ is non-degenerate for all n and t, then a good approximate solution to the time-dependent Schrödinger equation with initial condition $|\psi(0)\rangle = |n(0)\rangle$ is

$$|\psi(t)\rangle = \exp\left[\frac{-\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t'\, E_n(t')\right] \exp\{\mathrm{i}\gamma_n(t)\}|n(t)\rangle \tag{2.3}$$

where the factor in brackets is the familiar dynamical phase, and

$$\gamma_n(t) = \mathrm{i}\int_0^t \left\langle n(t)\left|\frac{\mathrm{d}}{\mathrm{d}t}\right| n(t)\right\rangle \mathrm{d}t. \tag{2.4}$$

Berry's remarkable observation is that in general the phase $\gamma_n(t)$ is non-integrable or anholonomic, for in general

$$\gamma_n(\tau) \neq \gamma_n(0). \tag{2.5}$$

The phase $\gamma_n(\tau)$ is commonly known as the Berry's phase. Berry showed that $\gamma_n(\tau)$ is gauge invariant in the sense that it is independent of the choice of the eigenvector associated with the non-degenerate eigenvalue $E_n(t)$, provided that one requires single-valuedness for the chosen eigenvector.

There have been many generalizations of Berry's pioneering work, removing the adiabatic condition and the condition of non-degeneracy. Aharonov and Anandan [5] devised a generalization which dispensed with any consideration of the periodic path in *classical* parameter space. They considered a cyclic path in the projective space $\mathscr{P}(\mathscr{H})$ associated with the Hilbert space $\mathscr{H}$, and showed that the Berry phase $\gamma_n(\tau)$ defined by (2.1), (2.2), (2.3), and (2.4) is a special case of the geometric phase with the cycle in projective space defined by the solution to the time-dependent Schrödinger equation under some specified conditions. One may also regard the parameters of the Hamiltonian as coordinates in the projective Hilbert space, thus labelling quantum states. This procedure is completely quantum mechanical. Furthermore, Berry's adiabaticity requirement is no longer needed. It has been shown rigorously for certain model systems that the Berry phase can be recovered from the Aharonov–Anandan phase in appropriate limits [7, 8].

There have been numerous attempts to generalize the Aharonov–Anandan phase so as to define the geometric phase for open paths in the projective space $\mathscr{P}(\mathscr{H})$. We shall not review the history of these proposals, but shall merely accept the prescription of Aitchison and Wanelik [19], which has all desirable features. Let $C:[t_1, t_2]\rightarrow\mathscr{P}(\mathscr{H})$, so that an unnormalized vector $|\Psi(t)\rangle \in \mathscr{H} - \{0\}$ may be considered as tracing an open path on the interval $[t_1, t_2]$. Then the *manifestly* gauge-invariant geometric phase $\xi(C)$ for the path C (whether open or closed) defined by Aitchison and Wanelik is a functional

$\xi(C) \equiv \xi[|\Psi(t)\rangle]$ satisfying

$$\exp(\mathrm{i}\xi[|\Psi(t)\rangle]) = \left[\left(\frac{\langle\Psi(t_1)|\Psi(t_2)\rangle}{\langle\Psi(t_2)|\Psi(t_1)\rangle}\right)\right]^{1/2} \exp\left[-\int_{t_1}^{t_2} \left\langle \frac{\Psi(t)}{\|\Psi(t)\|} \middle| \frac{\mathrm{d}}{\mathrm{d}t} \middle| \frac{\Psi(t)}{\|\Psi(t)\|} \right\rangle \mathrm{d}t\right]. \qquad (2.6)$$

It is easily verified that $\xi[|\Psi(t)\rangle]$ has the following desired properties: (1) it is always real; (2) it is a purely geometric quantity in that its value is reparameterization invariant or, in other words, it is independent of the speed with which the path $|\Psi(t)\rangle$ is traced out; and most importantly (3) it is projective-geometric in nature, meaning that it is the same for all possible paths $|\Psi(t)\rangle$ which project to a given path in the projective Hilbert space $\mathscr{P}(\mathscr{H})$. This last property states that

$$\xi[|\Psi(t)\rangle] = \xi[Z(t)|\Psi(t)\rangle] \qquad (2.7)$$

where $Z(t)$ is any sufficiently smooth C^*-valued function of t, C^* being the non-zero complex numbers. Thus the definition (2.6) for the (cyclic or non-cyclic) geometric phase $\xi(C)$ is justified. As shown by Aitchison and Wanelik, their prescription of the non-cyclic geometric phase is equivalent to the earlier but more complicated prescription (requiring attention to geodesics in projective space) of Samuel and Bhandari [15].

3. The Tomita–Chiao Setup

Tomita and Chiao [11] have observed the geometric phase for a single beam of photons passing through a single-mode, helically wound optical fibre. Their experimental setup,

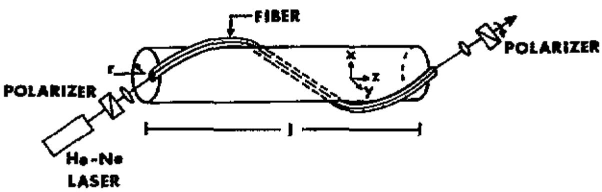

Figure 1. The Tomita–Chiao optical arrangement to measure the adiabatic geometrical phase.

schematically shown in figure 1, utilizes the adiabatic invariance of the helicity $\hat{\boldsymbol{s}} \cdot \boldsymbol{k}$ of the massless spin-1 boson, where $\hat{\boldsymbol{s}}$ is the spin operator and $\boldsymbol{k}$ is the direction of its propagation. Unlike a massive spin-1 boson, the masslessness of the photon guarantees that in the adiabatic limit its helicity remains either $+1$ or -1. Consequently, the spin of the photon always follows the direction $\boldsymbol{k}$ of its propagation. This allows Tomita and Chiao to parallel Berry's adiabatic treatment of spin-$\frac{1}{2}$ system in a slowly changing magnetic field to obtain the geometric phase for a beam of photons passing through an environment which adiabatically changes the direction $\boldsymbol{k}$. As discussed by Chiao and Wu [20], there are at least three cases in which $\boldsymbol{k}$ can change adiabatically: (1) when circularly polarized light propagates down a helically wound optical fibre; (2) when linearly polarized light propagates down such a fibre; and (3) when microwaves propagate down a helically wound circular waveguide. In all three cases there should be no sharp kinks in the fibre or waveguide, on the scale of a wavelength, so that the helicity of the photon does not flip sign as it propagates. Furthermore, any linear birefringence in the medium, and any ellipticity in the cross-sectional shape of the waveguide, which can cause conversion between states of opposite helicity, are neglected. The conditions for adiabaticity are discussed in [20], which include L, $R_c \gg d$ where L is the length, R_c the radius of curvature, and d the cross-sectional diameter of the fibre.

Let us assume that the light propagates inside a twisting waveguide in a single model [11] and its optical path is parameterized by t. The adiabatic invariance of the helicity of the photon implies that at each instant t, the photon's spin state satisfies

$$\hat{s}\cdot \boldsymbol{k}(t)|\boldsymbol{k}(t), \sigma\rangle = \sigma|\boldsymbol{k}(t), \sigma\rangle \tag{3.1}$$

where $\sigma = \pm 1$ is the time-independent helicity quantum number of the photon and $\boldsymbol{k}(t)$ is its direction of propagation at time t. Equation (3.1) is formally identical to the well known example of Berry [4] for a spin s in an adiabatically changing magnetic field $\boldsymbol{B}(t)$

$$g\hat{s}\cdot \boldsymbol{B}(t)|\boldsymbol{B}(t), m_s\rangle = E|\boldsymbol{B}(t), m_s\rangle \tag{3.2}$$

where g is related to the gyromagnetic ratio and m_s is the spin component along the direction of $\boldsymbol{B}(t)$.

In the actual experiment, an optical-fibre of fixed length is wound onto a cylinder of a fixed radius to form a helix as shown in figure 1. The pitch angle θ, which is also the angle between local waveguide and the helix axes in momentum space, traces out a curve C on the surface of a sphere corresponding to the fibre path. The Berry phase, in the present case of a massless spin-1 particle, may be obtained by considering a time-dependent problem in momentum space. As long as we stick to the momentum space of the photon, and as long as there are no sources in the vicinity, we can consider the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{k}(t)\rangle = \hat{H}(t)|\boldsymbol{k}(t)\rangle \tag{3.3}$$

in the *usual quantum mechanical sense* [21], where $\hat{H}(t)$ is the Hamiltonian operator. The evolution of the spin of the photon is governed by a Hamiltonian

$$\hat{H}(t) = \hat{H}_0 + \lambda\hat{s}\cdot \boldsymbol{k}(t) \tag{3.4}$$

where λ is the coupling constant related to the optical activity coefficient of the fibre and $\hat{H}_0|\boldsymbol{k}(t)\rangle = E_0|\boldsymbol{k}(t)\rangle$ defines the background propagation [20] (also see [8] for derivation of this Hamiltonian). This is the most general Hamiltonian which can be formed from the two vectors $\hat{s}$ and $\boldsymbol{k}(t)$ in the waveguide for a massless spin-1 particle in an isotropic medium with an isotropic cross-sectional boundary. The wavevector $\boldsymbol{k}(t)$, according to figure 1, can be expressed in Cartesian coordinates as

$$\boldsymbol{k}(t) = k[\hat{x}\sin\theta\cos\phi(t) + \hat{y}\sin\theta\sin\phi(t) + \hat{z}\cos\theta]. \tag{3.5}$$

The Hamiltonian of (3.4) can now be written as

$$\hat{H}(\phi(t)) = \hat{H}_0 + \lambda\hat{s}\cdot \boldsymbol{k}(\phi(t)) \tag{3.6}$$

with varying parameter ϕ. Then, following Berry [14], the excursion of the system between times $t=0$ and $t=\tau$ can be pictured as transport round a closed path $\phi(t)$ in parameter space, with Hamiltonian $\hat{H}(\phi(t))$ and such that $\phi(\tau) = \phi(0)$. In what follows, we take $\phi(0)=0$ and $\phi(\tau)=2\pi$. For the adiabatic approximation to apply, τ must be large. The state-vector $|\boldsymbol{k}(t)\rangle$ of the system now evolves according to the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{k}(t)\rangle = \hat{H}(\phi(t))|\boldsymbol{k}(t)\rangle. \tag{3.7}$$

At any instant, the natural basis consists of eigenstates $|E_n(\phi)\rangle$ of $\hat{H}(\phi)$ for $\phi=\phi(t)$, that satisfy

$$\hat{H}(\phi)|E_n(\phi)\rangle = E_n(\phi)|E_n(\phi)\rangle \tag{3.8}$$

with energies $E_n(\phi)$. Then, for a complete cycle c, Berry's phase in the present case of a photon is shown in the appendix to be

$$\gamma_\pm(c) = \mp 2\pi(1-\cos\theta) \tag{3.9}$$

where the subscripts $\pm$ specify eigenstates of + or − helicities. As discussed above, we have excluded $n=0$ as a consequence of masslessness of photon.

Next, we modify the above arrangement to exhibit geometric phases for non-cyclic variations of quantum state vectors. We consider a more general winding in which the winding angle ϕ is $<2\pi$, so that the photons are propagated through only a portion of the helix. This implies that the path in parameter (momentum) space is open, and the geometric phase acquired by the state-vector of the photon is of a non-cyclic type. As discussed in section 2, there are several formulations of the non-cyclic geometric phase equivalent to that of Aitchison and Wanelik [29] expressed in (2.6), but theirs belongs to the set of manifestly gauge-invariant prescriptions. Using this general definition we obtain the geometric phase difference $\gamma(\phi(t))$ between the initial state-vector $|\boldsymbol{k}(0)\rangle$ and the adiabatic state-vector $|\boldsymbol{k}(t)\rangle = \mathrm{e}^{\mathrm{i}\alpha(\phi(t))}|E_n(\phi(t))\rangle$ at an arbitrary point $\phi(t)$ on the Tomita–Chiao optical fibre to be

$$\exp[\mathrm{i}\gamma(\phi(t))] = \sqrt{\frac{\langle\boldsymbol{k}(\phi(0))|\boldsymbol{k}(\phi(t))\rangle}{\langle\boldsymbol{k}(\phi(t))|\boldsymbol{k}(\phi(0))\rangle}}\exp\left[-\int_0^t\left\langle\boldsymbol{k}(\phi(t'))\left|\frac{\mathrm{d}}{\mathrm{d}t'}\right|\boldsymbol{k}(\phi(t'))\right\rangle\mathrm{d}t'\right]. \tag{3.10}$$

It is worth re-emphasizing that the geometric phase given by the expression (3.10) for the open path $|\boldsymbol{k}(\phi(t))\rangle$ on the interval $[\phi(0), \phi(t)]$ is *manifestly* gauge invariant. Upon using $|\boldsymbol{k}(\phi(t))\rangle = \mathrm{e}^{\mathrm{i}\alpha(\phi(t))}|E_n(\phi(t))\rangle$, the expression (3.10) immediately yields

$$\gamma_n(\phi(t)) = \delta_n(\phi(t)) + \mathrm{i}\int_0^t\left\langle E_n(\phi(t'))\left|\frac{\mathrm{d}}{\mathrm{d}t'}\right|E_n(\phi(t'))\right\rangle\mathrm{d}t' \tag{3.11}$$

where

$$\exp[\mathrm{i}\delta_n(\phi(t))] \equiv \left[\frac{\langle E_n(\phi(0))|E_n(\phi(t))\rangle}{\langle E_n(\phi(t))|E_n(\phi(0))\rangle}\right]^{1/2}. \tag{3.12}$$

Using reparametrization invariance of (3.10), the geometric phase of (3.11) can equivalently be written as

$$\gamma_n(\phi) = \delta_n(\phi) + \mathrm{i}\int_0^\phi\left\langle E_n(\phi')\left|\frac{\mathrm{d}}{\mathrm{d}\phi'}\right|E_n(\phi')\right\rangle\mathrm{d}\phi' \tag{3.13}$$

with $\phi(0)=0$. As obtained in the appendix, this phase is explicitly expressed as

$$\gamma_{\pm}(\theta,\phi)=\delta_{\pm}(\theta,\phi)\pm\phi\cos\theta\mp\tan^{-1}(\cos\theta\tan\phi) \tag{3.14}$$

with

$$\delta_{\pm}(\theta,\phi)\equiv\pm\tan^{-1}\left[\frac{\sin^2\theta\cos\theta\sin\phi(1-\cos\phi)}{1+\cos^2\theta+\sin^2\theta(\cos\phi-\sin^2\phi)}\right] \tag{3.15}$$

obtained by using the eigenkets $|E(\phi)\rangle$ of (A.4) from the appendix. Equation (3.14) reduces to (3.9) for cyclic evolutions as expected.

At this juncture it should be noted that non-cyclic geometric phases in the framework of Samuel and Bhandari [15] have already been observed in various physical situations. For instance, Weinfurter and Badurek have observed the geometric phase effects for non-cyclic evolutions of thermal neutrons [22]. The experiment of Kwon *et al* is of particular relevance in the context of linearly polarized photons [23]. They injected a beam of linearly polarized photons down a uniformly wound half-turn single-mode optical fibre with various pitch angles, just as in the original Tomita–Chiao experiment. However, the use of the half-turn helix (which means $\phi=\pi$ in our notation) allowed them to observe the phases for open cycles. Their experiment, however, is subject to the same objections as that raised by Berry [12] and others [13] regarding the original Tomita–Chiao experiment; i.e. the phases they observed can be understood most naturally in terms of *classical* electromagnetism, and a quantum mechanical description of the rotation of the plane of polarization in their experiment is not necessary. Furthermore, even before the Tomita–Chiao experiment, rotation of the plane of polarization of linearly polarized light travelling down a single-mode optical fibre (with or without parallel ends, i.e. with or without a complete cycle) was studied by Ross [24] and Haldane [25] at the classical level and shown to agree with experiments. Thus, the experiment of Kwon *et al* has no conceptual improvement over the previous works of Ross and Haldane.

To ensure gauge-invariance and actually to compute the geometric phases for open paths, Kwon *et al*, used the 'geodesic rule' of Samuel and Bhandari [15]. We, on the other hand, use the conceptually more clear and manifestly gauge-invariant prescription of Aitchison and Wanelik, which naturally incorporates the geodesic rule. As a matter of fact, the equivalence of the Samuel–Bhandari method with the prescription of Aitchison and Wanelik has been shown by various authors [19]. Therefore, it is not surprising that the results of Kwon *et al* for the half-turn (their equation (1)) and the quarter-turn (their equation (2)) fibres agree with our (3.14) above, for the physical range $0\leqslant\theta\leqslant\pi/2$ of the pitch angle, upon substitutions of $\phi=\pi$ and $\phi=\pi/2$. These substitutions must be made because they consider only the half-turn and the quarter-turn fibres, whereas our formula (3.14) is for any ϕ, ie. for any arbitrary interval on the fibre.

The above considerations, then, give us strong reasons to stress the quantum mechanical aspect of our proposal below as well as the fact that we use a conceptually superior prescription to obtain the non-cyclic geometric phases.

4. Review of two-photon interferometry

Let us consider photon pairs produced by parametric down-conversion, a process in which a single photon incident upon a crystal gives rise to a pair of non-locally correlated photons [26]. We use the general arrangement for two-photon interferometry proposed by HSZ [18] in which an ensemble of particle pairs is emitted by a source into the beams A, B, C, D, with wavevectors $\boldsymbol{k}_A$, $\boldsymbol{k}_B$, $\boldsymbol{k}_C$, and $\boldsymbol{k}_D$, satisfying

$$\boldsymbol{k}_A+\boldsymbol{k}_C=\boldsymbol{k}_D+\boldsymbol{k}_B=\boldsymbol{k} \tag{4.1}$$

where $\boldsymbol{k}$ is the wavevector of the incident beam. Each pair in the ensemble is in the quantum state

$$|\Psi\rangle=\frac{1}{\sqrt{2}}\{|\boldsymbol{k}_A\rangle_1\otimes|\boldsymbol{k}_C\rangle_2+|\boldsymbol{k}_D\rangle_1\otimes|\boldsymbol{k}_B\rangle_2\} \tag{4.2}$$

where $|\boldsymbol{k}_A\rangle_1$, $|\boldsymbol{k}_C\rangle_2$, $|\boldsymbol{k}_D\rangle_1$, $|\boldsymbol{k}_B\rangle_2$, are the approximate eigenvectors of linear momentum operators, and

$$|\boldsymbol{k}_A|=|\boldsymbol{k}_D| \qquad |\boldsymbol{k}_B|=|\boldsymbol{k}_C|. \tag{4.3}$$

The detailed arrangement is shown in figure 2 in which U_i and V_i are detectors, with index $i=1,2$ labelling the particle that is registered in the respective detector.

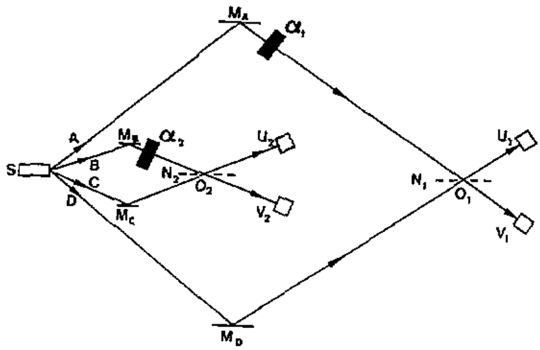

Figure 2. The arrangement of HSZ for two-photon interferometry with variable phase shifters.

The state $|\Psi\rangle$ in (4.2) describes a coherent superposition of two distinct pairs of correlated paths for particles 1 and 2: (I) particle 1 in beam A and particle 2 in beam C, and (II) particle 1 in beam D and particle 2 in beam B. In the pair (I), particle 1 in beam A is reflected from mirror M_A to phase shifter α_1 en route to half-silvered mirror N_1, from which it proceeds either to detector U_1 or to detector V_1; while particle 2 in beam C is reflected from mirror M_C to half-silvered mirror N_2, from which it proceeds to detectors U_2 or V_2. In the pair (II), on the other hand, particle 1 in beam D proceeds to U_1 or V_1 via M_D and N_1, while particle 2 in beam B proceeds to U_2 or V_2 via M_B, α_2, and N_2. Thus, the beams A and D of particle 1 are given a variable relative phase shift α_1 before recombination near the point O_1 on the beam splitter N_1, whereas the beams B and C of particle 2 are given a variable relative phase shift α_2 before recombination near the point O_2 on the beam splitter N_2. From (4.3) it follows that the frequencies of the two interfering beams at N_1 are equal, and the frequencies of the two interfering beams at N_2 are equal, even though the frequencies at N_1 and N_2 are unequal. The observed quantities of interest are the two-photon coincident count rates, as functions of relative phases α_1 and α_2, predicted by quantum mechanics. These quantum-mechanical probabilities for joint detection of particles 1 and 2 are proportional to the absolute square of the total amplitudes obtained by the superpositions of the amplitudes associated with each of the two pairs of correlated paths. For example, if the detectors have quantum efficiency η then the quantum-mechanical probability for joint detection of particles 1 and 2 by detectors U_1 and U_2, when phase shifts α_1 and α_2 have been chosen, is η^2 times the absolute square of the total amplitude:

$$A(U_1, U_2|\alpha_1, \alpha_2) = \frac{1}{\sqrt{2}}\left[\left(\frac{\mathrm{i}}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\alpha_1}\right)_1\left(\frac{1}{\sqrt{2}}\right)_2 + \mathrm{e}^{\mathrm{i}\beta}\left(\frac{1}{\sqrt{2}}\right)_1\left(\frac{\mathrm{i}}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\alpha_2}\right)_2\right]. \tag{4.4}$$

Here, the factors $\mathrm{e}^{\mathrm{i}\alpha_1}$ and $\mathrm{e}^{\mathrm{i}\alpha_2}$ arise from the phase shifters encountered along the respective paths, and the factors $\mathrm{i}/\sqrt{2}$ and $1/\sqrt{2}$ arise [27], respectively, from reflection and transmission at the beam splitters. The subscripts 1 and 2 on the parentheses in (4.4) refer to a path of particle 1 and the correlated path of particle 2, respectively. The phase factor $\mathrm{e}^{\mathrm{i}\beta}$ depends upon the detailed placement of the mirrors and beam splitters and is independent of α_1 and α_2. Using (4.4) and analogous equations for the amplitudes $A(V_1, V_2|\alpha_1, \alpha_2)$, $A(U_1, V_2|\alpha_1, \alpha_2)$, and $A(V_1, U_2|\alpha_1, \alpha_2)$, the quantum-mechanical probabilities for the joint detection by the detector pairs (U_1, U_2), (V_1, V_2), (U_1, V_2), and (V_1, U_2), respectively, can be obtained as η^2 times the absolute squares of these amplitudes. The results are

$$\begin{aligned} P(U_1, U_2|\alpha_1, \alpha_2) &= P(V_1, V_2|\alpha_1, \alpha_2) \\ &= \eta^2\left[\frac{1}{4} + \frac{1}{4}\cos(\alpha_2 - \alpha_1 + \beta)\right] \end{aligned} \tag{4.5a}$$

and

$$\begin{aligned} P(U_1, V_2|\alpha_1, \alpha_2) &= P(V_1, U_2|\alpha_1, \alpha_2) \\ &= \eta^2\left[\frac{1}{4} - \frac{1}{4}\cos(\alpha_2 - \alpha_1 + \beta)\right]. \end{aligned} \tag{4.5b}$$

It is evident from the sinusoidal dependence of these probabilities that the interference fringes, the signature of quantum phenomena, can be exhibited by monitoring the coincidence count rates while varying the phase shifts α_1 and α_2. Note that, as emphasized by HSZ, only the *coincidence* count rate exhibits interference fringes since the count rate of each of the four detectors singly is constant, independent of α_1 and α_2:

$$\begin{aligned} P(U_1|\alpha_1, \alpha_2) &= P(V_1|\alpha_1, \alpha_2) \\ &= P(U_2|\alpha_1, \alpha_2) \\ &= P(V_2|\alpha_1, \alpha_2) = \frac{\eta}{2} \end{aligned} \tag{4.6}$$

which is a direct consequence of the entangled character of the state $|\Psi\rangle$.

5. Non-cyclic geometric phase in two-photon entangled state

Now, to obtain the effect of the geometric phases associated with the two subsystems, namely the photons 1 and 2, in the HSZ-arrangement discussed above, we propose to replace the phase shifters α_1 and α_2 by two Tomita–Chiao apparatus providing the geometric phase shifts $\gamma_{\pm}^{1} \equiv \gamma_{\pm}(\theta_1)$ and $\gamma_{\pm}^{2} \equiv \gamma_{\pm}(\theta_2)$. Here θ_1 and θ_2 are the colatitudes of the wavevector k in momentum spaces of photons 1 and 2, respectively, in the geometries of Tomita–Chiao apparatus discussed above; and the circularly polarized light with definite helicity + or − could be obtained by inserting quarter-wave plates q_A, q_B, q_C, q_D in the beams A, B, C, D of linearly polarized photons emerging from the source S of figure 2. These geometric phase effects will be seen in the probability expressions of equations (4.5a) and (4.5b). The final experimental arrangement is shown

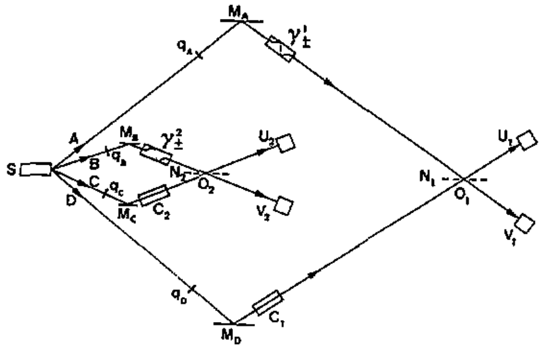

Figure 3. Final arrangement of the experimental setup to measure cyclic and non-cyclic geometric phases after replacing the phase shifters in the HSZ arrangement by the Tomita–Chiao apparatus and inserting quarter-wave plates q_A, q_B, q_C, q_D.

in figure 3. The dynamical phase factors of the two arms of the interferometers, which have equal optical path lengths, are the same, and do not enter into the coincident counts given by (4.5a) and (4.5b). Even if these dynamical phase factors are unequal and depend upon the detailed placement of the mirrors and beam splitters, they would not affect the changes of geometric phase differences. This can be easily seen by comparing (4.4) with (4.5a) and (4.5b), and observing that the phase factor $e^{i\beta}$ of (4.4), which depends upon the detailed placement of the mirrors and beam splitters, has only a trivial effect (namely a uniform shift) on the probabilities for the joint detections given by (4.5a) and (4.5b). On the other hand, however, the optical fibres C_1 and C_2 shown in figure 3 are quite essential to compensate for the path lengths of the helical fibres. The compensating fibre C_1 (C_2) is identical to that providing the phase shift $\gamma_{\pm}^{1}$ ($\gamma_{\pm}^{2}$) except that the cylinder on which it is wound has a vanishingly small radius compared to the radius of the cylinder on which the fibre producing the phase shift $\gamma_{\pm}^{1}$ ($\gamma_{\pm}^{2}$) is wound. The vanishingly small radius guarantees almost zero solid angle subtended by the k-vector in the momentum space implying negligible geometric phase shifts. In the non-cyclic case, unlike the cyclic case, the geometry of the final arrangement shown in figure 3 has to be slightly modified because, in general, the directions of the input and output beams for the Tomita–Chiao apparatus will not be the same.

It might appear, at first sight, that our usage of the Berry's formalism to obtain the geometric phase for a subsystem of an entangled composite system is inadequate because the usual Berry phase is defined only for *pure* quantum-mechanical states, whereas the subsystem by itself is in an indefinite or mixed state. This is not the case, however. Since there is no interaction among the subsystems of the considered two-photon entangled composite system, the total Hamiltonian $H_{tot}(t)$ for the composite system can always be split as

$$H_{\mathrm{tot}}(t) = H_1(t) \oplus \mathbf{1} + \mathbf{1} \oplus H_2(t) \tag{5.1}$$

where $H_j(t)$, $j=1, 2$, is the Hamiltonian of the subsystem j alone and is an operator on the Hilbert space corresponding to that subsystem. This splitting of the total Hamiltonian allows us to write the Schrödinger-type equation

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\boldsymbol{k}_l(t)\rangle_j = H_j(t)|\boldsymbol{k}_l(t)\rangle_j \tag{5.2}$$

for *each* of the eigenvectors $|\boldsymbol{k}_l(t)\rangle_j$ ($l=A, D$ if $j=1$, $l=B, C$ if $j=2$) of the entangled state of (4.2), rendering our usage of the usual Berry's formalism adequate. The dynamical equations (5.2) follow from the evolution operator for the composite system:

$$\begin{aligned} U(t) &\equiv T\exp\left[-\mathrm{i}\int_0^t H_{\mathrm{tot}}(t')\,\mathrm{d}t'\right] \\ &= T\exp\left[-\mathrm{i}\int_0^t H_1(t')\,\mathrm{d}t'\right] \otimes T\exp\left[-\mathrm{i}\int_0^t H_2(t')\,\mathrm{d}t'\right] \end{aligned} \tag{5.3}$$

where T stands for time-ordering. This factorizability of the time evolution operator can be checked by series expansion. Now the composite system is represented by the pure quantum state

$$|\Psi(t_0)\rangle = \frac{1}{\sqrt{2}}\,\{|\boldsymbol{k}_A(t_0)\rangle_1 \otimes |\boldsymbol{k}_C(t_0)\rangle_2 + |\boldsymbol{k}_D(t_0)\rangle_1 \otimes |\boldsymbol{k}_B(t_0)\rangle_2\} \tag{5.4}$$

as in (4.2) of section 4. Then, from (5.3) and (5.4) we obtain a vector representing the state of the composite system at an arbitrary time t:

$$\begin{aligned} |\Psi(t)\rangle &= U(t-t_0)|\Psi(t_0)\rangle \\ &= \frac{1}{\sqrt{2}}\left[\mathrm{T}\exp\left\{-\mathrm{i}\int_0^t H_1(t'-t_0)\,\mathrm{d}t'\right\}|\boldsymbol{k}_A(t_0)\rangle_1\right. \\ &\quad \otimes T\exp\left\{-\mathrm{i}\int_0^t H_2(t'-t_0)\,\mathrm{d}t'\right\}|\boldsymbol{k}_C(t_0)\rangle_2 \\ &\quad + T\exp\left\{-\mathrm{i}\int_0^t H_1(t'-t_0)\,\mathrm{d}t'\right\}|\boldsymbol{k}_D(t_0)\rangle_1 \\ &\quad \left.\otimes T\exp\left\{-\mathrm{i}\int_0^t H_2(t'-t_0)\,\mathrm{d}t'\right\}|\boldsymbol{k}_B(t_0)\rangle_2\right]. \end{aligned} \tag{5.5}$$

We conclude this section by emphasizing that the possibility of unequivocal observation of *non-cyclic* geometric phases for individual photons propagating in twisted dielectrics is the most novel feature of our proposed arrangement. As far as we know, no such experiment at the quantum mechanical level has yet been carried out. Consequently, here we have essentially a novel prediction which deserves to be experimentally tested.

6. Discussion: Non-classical nature of the geometric phases

In this paper we have successfully answered Berry's reservations [12] regarding the experiment of Tomita and Chiao to measure geometric phases for photons, and have demonstrated the truly quantum-mechanical geometric phase effects associated with correlated photons spiraling around helical fibres. That the geometric phases obtained above are truly quantum-mechanical in nature is exhibited by (4.5) and (4.6), which express the non-classical, entangled character of the two-photon state. Moreover, a Bell-type inequality can be derived for these arrangements [28], which puts classical probabilities in conflict with (4.5) and (4.6). After the insertion of the Tomita–Chiao phase shifters in the HSZ-arrangement, these equations of the quantum-mechanical probabilities for the joint detection read

$$P\left(U_1,\ U_2 \mid \gamma^1_\pm,\ \gamma^2_\pm\right)=P\left(V_1,\ V_2 \mid \gamma^1_\pm,\ \gamma^2_\pm\right)$$
$$=\eta^2\left[\frac{1}{4}+\frac{1}{4}\cos\left(\gamma^2_\pm-\gamma^1_\pm+\beta\right)\right] \tag{6.1a}$$

$$P\left(U_1,\ V_2 \mid \gamma^1_\pm,\ \gamma^2_\pm\right)=P\left(V_1,\ U_2 \mid \gamma^1_\pm,\ \gamma^2_\pm\right)$$
$$=\eta^2\left[\frac{1}{4}-\frac{1}{4}\cos\left(\gamma^2_\pm-\gamma^1_\pm+\beta\right)\right] \tag{6.1b}$$

and

$$P\left(U_1 \mid \gamma^1_\pm,\ \gamma^2_\pm\right)=P\left(V_1 \mid \gamma^1_\pm,\ \gamma^2_\pm\right)$$
$$=P\left(U_2 \mid \gamma^1_\pm,\ \gamma^2_\pm\right)$$
$$=P\left(V_2 \mid \gamma^1_\pm,\ \gamma^2_\pm\right)=\frac{\eta}{2}. \tag{6.2}$$

As emphasized by HSZ, this phenomenon of constant count rate of each of the four detectors singly is a direct consequence of the entangled character of the state of the two-photon composite system. Furthermore, it can be shown that entanglement guarantees a violation of a Bell inequality, and hence of classical probabilities [29].

The above discussion is, then, sufficient to render Berry's reservations [12] irrelevant for the experimental procedure proposed here to measure geometric phases. Berry's essential point is that the geometric phase effects associated with the optical rotations of photon fields can be most appropriately described at the level of classical eletromagnetism. Although this is true in most of the optical experiments performed so far to measure geometric phases (with the exception of the Kwiat and Chiao experiment [14]), it is not possible to give a classical description of the experimental procedure proposed here due to the entangled character of the considered composite system. Incidentally, Silverman has examined the Aharonov–Bohm effect in two solenoids with correlated particles [30] which has some similarity with our proposal. He, however, deals with only closed paths for charged particles in the physical space, whereas we consider *partial* cycles in the momentum space of photons.

A clarification of a potentially confusing point is in order here. Our experimental arrangement proposes to test the non-cyclic geometric phase *not* for a *single* Tomita–Chiao helix but for two of them in tandem, and what enters into the probability expressions (6.1a) and (6.1b) is the *difference* between the two geometric phases. Does this weaken our test? Could it be that the difference between two phases is really a cyclical phase in disguise? The answer to both questions is no. The use of two helices is *not* essential for the argument. In fact, one can make the geometric phase due to one of the helices negligible (like the compensators), or even replace it by another type of phase shifter (like a variable glass plate.) The usage of two Tomita–Chiao helices instead of only one is for the generality and symmetry of the proposed arrangement. It allows freedom and convenience in the realization of the experiment. It is true that what would ultimately be measured is a *relative* phase between the two arms of the interferometer. But the point is that the geometric phase changes which appear in the expressions for the joint detection probabilities are generated by *non-cyclic* variations in the parameter (momentum) space of the subsystem of the composite system.

In addition to giving strictly quantum-mechanical geometric phases, as we saw in section 5, the general arrangement we have proposed here also allows us unambiguously to obtain gauge-invariant *non-cyclic* phases for entangled photon states. The prediction of this effect for entangled subsystems is the central thesis of the present paper and provides an experimental test for non-cyclic geometric phases. To our knowledge there has not been so far any test of a strictly quantum mechanical prediction of a non-cyclic geometric phase for photons.

Acknowledgements

One of the authors (J C) would like to thank Wolfson College, Oxford for the Research Fellowship which partially supported this work, and Dr I J R Aitchison, Dr H Brown, and K Wanelik for discussions. The authors are grateful to Dr M A Horne and Dr A Zeilinger for stimulating discussions. A S wishes to thank the National Science Foundation for support of this research under grant PHY9022345.

Appendix. Calculations of cyclic and non-cyclic Berry phases

In this appendix we shall explicitly obtain the Berry phase given by (3.9) for cyclic evolution of photons in a helical fibre, as well as the non-cyclic version of such phase given by (3.14).

The most general Hamiltonian which can be formed from the two vectors $\hat{s}$ and $\boldsymbol{k}(t)$ in the considered optical waveguide for a massless spin-1 particle is

$$\hat{H}(\phi) = \hat{H}_0 + \lambda \hat{s} \cdot \boldsymbol{k}(\phi), \tag{A.1}$$

as in (3.8). Now, let us look for instantaneous eigenvalues and eigenvectors of this Hamiltonian:

$$\hat{H}(\phi) | E(\phi) \rangle = E(\phi) | E(\phi) \rangle$$

with

$$\hat{H}_0 = E_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\hat{H}_0$ defines the background propagation. If we use the representations

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & +\mathrm{i} & 0 \end{pmatrix} \quad S_y = \begin{pmatrix} 0 & 0 & +\mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ +\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.2}$$

for the spin-1 operators and $\boldsymbol{k}(\phi)$ from (3.5), then the eigenvalue problem can be written as

$$\begin{pmatrix} E_0 & -\mathrm{i}\lambda k \cos\theta & \mathrm{i}\lambda k \sin\theta \sin\phi \\ \mathrm{i}\lambda k \cos\theta & E_0 & -\mathrm{i}\lambda k \sin\theta \cos\phi \\ -\mathrm{i}\lambda k \sin\theta \sin\phi & \mathrm{i}\lambda k \sin\theta \cos\phi & E_0 \end{pmatrix} \begin{pmatrix} \psi_x(\phi) \\ \psi_y(\phi) \\ \psi_z(\phi) \end{pmatrix} = E(\phi) \begin{pmatrix} \psi_x(\phi) \\ \psi_y(\phi) \\ \psi_z(\phi) \end{pmatrix}. \tag{A.3}$$

The eigenvalues are 0, $E_0 + \lambda k$, and $E_0 - \lambda k$, and the eigenvectors for $E(\phi) = E_0 \pm \lambda k$ are

$$|E(\phi) = E_0 \pm \lambda k\rangle \sim \frac{1}{(2 - 2\sin^2\theta \sin^2\phi)^{1/2}} \begin{pmatrix} \mp\cos\theta + \mathrm{i}\sin^2\theta \sin\phi \cos\phi \\ \mathrm{i}\,(\sin^2\theta \sin^2\phi - 1) \\ \pm\sin\theta\cos\phi + \mathrm{i}\sin\theta\cos\theta\sin\phi \end{pmatrix}. \tag{A.4}$$

Here we have neglected the eigenvalue $E(\phi) = 0$ and the corresponding eigenvector as a consequence of masslessness of photon discussed in the text. From these eigenvectors one can compute

$$\left\langle E_\pm(\phi') \left| \frac{\mathrm{d}}{\mathrm{d}\phi'} \right| E_\pm(\phi') \right\rangle = \pm\mathrm{i}\left(\frac{\cos\theta \sin^2\theta \sin^2\phi'}{1 - \sin^2\theta \sin^2\phi'} \right). \tag{A.5}$$

This gives the geometric phase

$$\gamma_\pm(\phi) = \mathrm{i}\int_0^\phi \left\langle E_\pm(\phi') \left| \frac{\mathrm{d}}{\mathrm{d}\phi'} \right| E_\pm(\phi') \right\rangle \mathrm{d}\phi' = \mp\cos\theta \int_0^\phi \frac{\sin^2\phi'}{\left(\dfrac{1}{\sin^2\theta}\right) - \sin^2\phi'}\,\mathrm{d}\phi' \tag{A.6}$$

Now, for a complete cycle, $\phi=2\pi$, and to obtain the geometric phase in this case we must evaluate the definite integral

$$I=\int_0^{2\pi}\frac{\sin^2\phi'}{a^2-\sin^2\phi'}\,\mathrm{d}\phi' \qquad \left(\text{with } a=\frac{1}{\sin\theta}\right)$$

$$=\int_0^{2\pi}\left(\frac{a^2}{a^2-\sin^2\phi'}-1\right)\mathrm{d}\phi'$$

$$=-2\pi+\frac{a}{2}\int_0^{2\pi}\left(\frac{1}{a-\sin\phi'}+\frac{1}{a+\sin\phi'}\right)\mathrm{d}\phi'. \tag{A.7}$$

From the table of definite integrals ([31] item 15.43)

$$\int_0^{2\pi}\frac{\mathrm{d}x}{a\pm\sin x}=\frac{2\pi}{(a^2-1)^{1/2}}. \tag{A.8}$$

Therefore

$$I=-2\pi+\frac{2\pi}{\left(1-\frac{1}{a^2}\right)^{1/2}}. \tag{A.9}$$

Substituting $a=1/\sin\theta$ and rearranging yields

$$I=\frac{2\pi}{\cos\theta}(1-\cos\theta). \tag{A.10}$$

Hence, the geometric phase for a complete cycle is

$$\gamma_\pm(\theta)=\mp 2\pi(1-\cos\theta) \tag{A.11}$$

with θ as a pitch angle of the phase shifting helix.

For a partial cycle, $\phi\neq 2\pi$, we are faced with an indefinite integral

$$I(\phi)=\int_0^{\phi}\frac{\sin^2\phi'}{\left(\frac{1}{\sin^2\theta}\right)-\sin^2\phi'}\,\mathrm{d}\phi'=-\phi+a^2\int_0^{\phi}\frac{\mathrm{d}\phi'}{a^2-\sin^2\phi'} \tag{A.12}$$

with $a=1/\sin\theta$. Using the table of indefinite integrals ([31] item 14.363) we have

$$\int\frac{\mathrm{d}x}{a^2-\sin^2 x}=\frac{1}{a\sqrt{a^2-1}}\tan^{-1}\left(\frac{\sqrt{a^2-1}\,\tan x}{a}\right). \tag{A.13}$$

Substituting $a=1/\sin\theta$, evaluating from 0 to ϕ, and rearranging yields

$$I(\phi)=-\phi+\left(\frac{1}{\cos\theta}\right)\tan^{-1}(\cos\theta\tan\phi). \tag{A.14}$$

Hence, the geometric phase for non-cyclic evolution in the optical fibre is given by

$$\gamma_\pm(\theta,\phi)=\delta_\pm(\theta,\phi)\pm\phi\cos\theta\mp\tan^{-1}(\cos\theta\tan\phi) \tag{A.15}$$

as in (3.14) in the text, with $\delta_\pm(\theta,\phi)$ given by (3.15). Equation (A.15) reduces to (A.11) for cyclic evolutions as expected.

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